Int. J. Heat Mass Transfer. Vol. 35, No. 8, pp. 2057–2060, 1992
Printed in Great Britain
0017–9310/92 $5.00+0.00

Optimization of a thermally non-symmetric fin: preliminary evaluation

D. C. LOOK, JR. and H. S. KANG
Mechanical and Aerospace Engineering and Engineering Mechanics Department,
University of Missouri–Rolla, Rolla, MO 65401, U.S.A.

(*Received* 7 *January* 1991 *and in final form* 23 *July* 1991)

INTRODUCTION

IT HAS been shown that the one-dimensional approach is convenient, but may be in error for certain physical conditions [1–8]. Further, some papers [9–11] present the optimization of the fin heat transfer with respect to its geometry or its weight for the same one-dimensional conditions. All of these approaches are questionable when the convection coefficients of the fin surfaces are not equal and the root temperature is not constant.

This study produced an optimization procedure for the heat loss from a rectangular fin as a function of the fin ratio, L, when the root temperature is $T = T_w + a\cos^m(\pi y'/2\ell)$ and the convection coefficients, h_i, of all surfaces are constant but not equal (note that T_w and a are constant and the subscript, i, is 1—top, 2—bottom, and 3—tip). Of particular importance in this development is the relationship between the ratio L for $0.99(Q/k\theta_0)_{max}$ and the fin's usefulness. The analysis is based upon the usual assumptions [12] (i.e. constant properties, steady state, no heat sources, and Newton's law is valid). Finally, in describing the convection characteristics, the Biot number, B, will be used rather than the convection coefficient, with the restrictions that (1) $0 \leqslant B_2 \leqslant B_1 \leqslant 1$, and (2) $B_1 = 0.01$ and 1.0.

TWO-DIMENSIONAL ANALYSIS

For a two-dimensional rectangular fin with constant physical properties, the first law of thermodynamics for conduction is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \qquad (1)$$

where $\theta = T - T_\infty$, $\theta_0 = T_w - T_\infty$, $L = L'/\ell$, $x = x'/\ell$ and $y = y'/\ell$ (see Fig. 1).

The corresponding boundary conditions are, for all y

$$x = 0 \quad \theta = \theta_0 + a\cos^m\left(\frac{\pi y}{2}\right) \quad \text{and}$$

$$x = L \quad \frac{\partial\theta}{\partial x} + B_3\theta = 0 \qquad (2)$$

and, for all x

$$y = 1 \quad \frac{\partial\theta}{\partial y} + B_1\theta = 0 \quad \text{and}$$

$$y = -1 \quad \frac{\partial\theta}{\partial y} - B_2\theta = 0 \qquad (3)$$

where $B_i = h_i\ell/k$, $i = 1, 2, 3$ and k is the thermal conductivity. As described in ref. [8], the solution to equation (1) is

$$\theta = \sum f_1(y) f_2(x) N_{nm} \qquad (4)$$

and the heat lost per fin width in this two-dimensional case is

$$Q = \int_{-1}^{1}\left[-k\frac{\partial\theta}{\partial x}\right]_{x=0} \mathrm{d}y = -2k\theta_0 \sum_{n=1}^{\infty} f_n N_{nm} \sin(\lambda_n) \qquad (5)$$

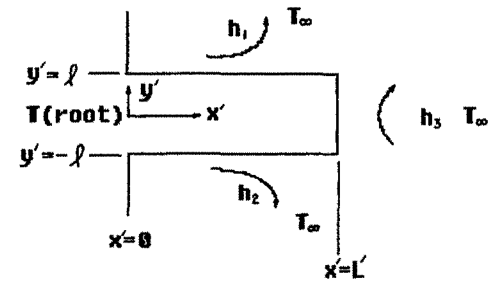

FIG. 1. Geometry of a thermally asymmetric, constant cross-sectional area, rectangular fin where $T(\text{root}) = T_w + a\cos^m(\pi y'/2\ell)$.

where

$$f_1(y) = \cos(\lambda_n y) + A_n \sin(\lambda_n y) \qquad (6)$$

$$f_2(x) = \cosh(\lambda_n x) + f_n \sinh(\lambda_n x) \qquad (7)$$

$$f_n = -\frac{B_3 + \lambda_n \tanh(\lambda_n L)}{\lambda_n + B_3 \tanh(\lambda_n L)} \qquad (8)$$

$$N_{n0} = D_n(1+b) \qquad (9)$$

$$N_{n1} = D_n\left[1 + \frac{2b\pi\lambda_n}{\pi^2 - 4\lambda_n^2}\cot(\lambda_n)\right] \qquad (10)$$

$$N_{n2} = D_n\left[1 + \frac{b\pi^2}{2(\pi^2 - \lambda_n^2)}\right] \qquad (11)$$

$$N_{nm} = D_n\left[1 + \frac{b\lambda_n\Gamma(m+1)}{2^m \sin(\lambda_n)\Gamma\left(\frac{m+2}{2} + \frac{\lambda_n}{\pi}\right)\Gamma\left(\frac{m+2}{2} - \frac{\lambda_n}{\pi}\right)}\right] \qquad (12)$$

$$D_n = \frac{\dfrac{2\sin(\lambda_n)}{\lambda_n}}{\left[\left(1 + \dfrac{2\sin(\lambda_n)}{2\lambda_n}\right) + A_n^2\left(1 - \dfrac{2\sin(\lambda_n)}{2\lambda_n}\right)\right]} \qquad (13)$$

$$A_n = \frac{\lambda_n \tan(\lambda_n) - B_1}{\lambda_n + B_1 \tan(\lambda_n)} = \frac{-\lambda_n \tan(\lambda_n) + B_2}{\lambda_n + B_2 \tan(\lambda_n)} \qquad (14)$$

and λ_n are the eigenvalues obtained from equation (14). The variations of this solution with the various parameters have been discussed previously [8] and will not be repeated here.

In order to determine the ratio L, which produces the limiting value of heat loss,

$$\frac{\mathrm{d}Q}{\mathrm{d}L} = 0. \qquad (15)$$

The result of this optimization is

$$\sum_{n=1}^{\infty} N_{nm} f d_n \sin(\lambda_n)\,\mathrm{sech}^2(\lambda_n L) = 0 \qquad (16)$$

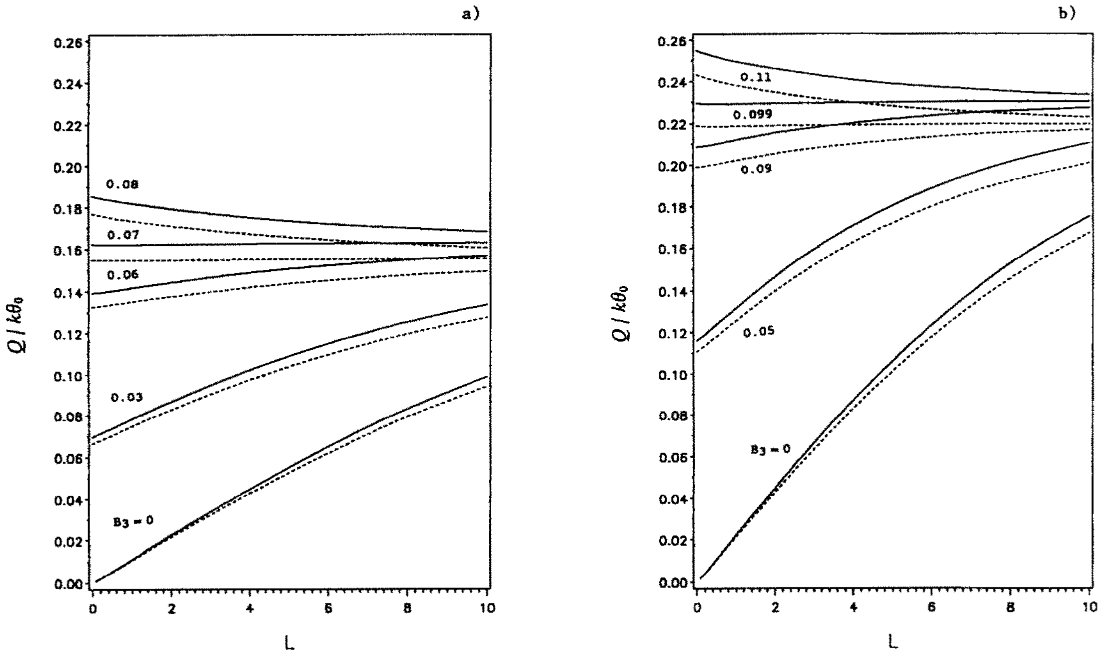

FIG. 2. Heat loss, $Q/k\theta_0$, from a fin vs the fin ratio, L, for $B_1 = 0.01$, and $b = 0.25$ for various values of B_3 and $m = 1$(——) and $m = 3$(- - - -): (a) $B_2 = 0$ and (b) $B_2 = 0.01$.

where

$$fd_n = \frac{\lambda_n(\lambda_n^2 - B_3^2)}{[\lambda_n + B_3 \tanh(\lambda_n L)]^2}. \tag{17}$$

Equation (16) will be satisfied as L approaches infinity; then $\tanh(\lambda_n L) \to 1$ and $f_n \to -1$ (see equation (8)). Thus, the maximum or minimum value of the heat loss from the fin is

$$Q_{\text{limiting}} = 2k\theta_0 \sum_{n=1}^{\infty} N_{nm} \sin(\lambda_n). \tag{18}$$

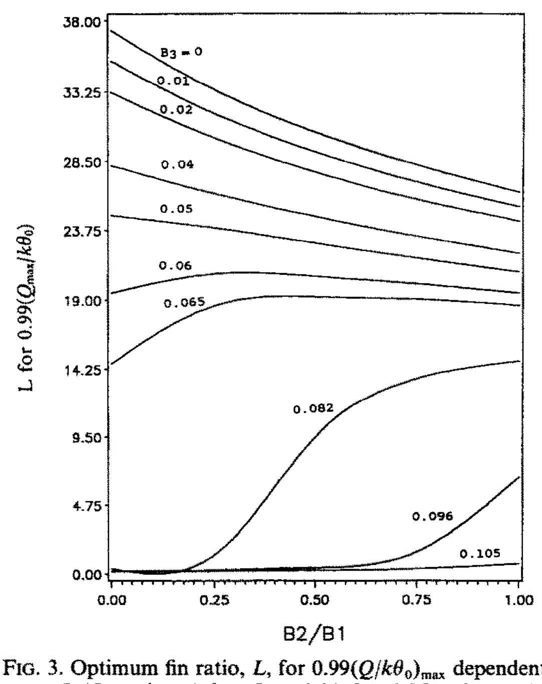

FIG. 3. Optimum fin ratio, L, for $0.99(Q/k\theta_0)_{\max}$ dependent upon B_2/B_1 and B_3 (when $B_1 = 0.01$, $b = 0.25$ and $m = 1$).

To determine whether the limiting Q is a maximum or minimum value, the second derivative is used. Then

$$\frac{d^2Q}{dL^2} = 2k\theta_0 \sum_{n=1}^{\infty} d_n(B_3^2 - \lambda_n^2) N_{nm} \sin(\lambda_n) \tag{19}$$

where

$$d_n = \frac{2\lambda_n^2 \operatorname{sech}^2(\lambda_n L)[B_3 \operatorname{sech}^2(\lambda_n L) + \lambda_n \tanh(\lambda_n L) + B_3 \tanh^2(\lambda_n L)]}{[\lambda_n + B_3 \tanh(\lambda_n L)]^3}. \tag{20}$$

The maximum value of Q as L approaches infinity ($Q_{\max}$) would occur when equation (19) is less than zero. Further, when this equation is greater than zero, $Q_{\min}$, occurs as L approaches infinity. A special case results for $L = 0$ (the 'no fin' case). For that condition $f_n = -B_3/\lambda_n$ and

$$Q_{\text{limiting}} = 2k\theta_0 \sum_{n=1}^{\infty} \frac{B_3}{\lambda_n} N_{nm} \sin(\lambda_n). \tag{21}$$

B_3, in this case, is interpreted as the coefficient of the wall, B_w.

RESULTS

Traditionally, from a one-dimensional analysis, a Biot number of 0.01 would indicate that a fin is more than justified. The results of this two-dimensional analysis for B_1 equal 0.01 are presented in Fig. 2. Note that in both cases the heat loss increases for small values of B_3 until B_3 reaches a certain value for which the heat loss is essentially independent of L. Above this value of B_3, the heat loss decreases as L increases. Also, as the value of B_2 increases, the value of B_3, which makes the heat loss decrease as the L increases, becomes larger. Finally, these figures show that the difference between $Q/k\theta_0$ ($m = 1$) and $Q/k\theta_0$ ($m = 3$) increases as L increases but the curves have the same trend.

An alternate procedure for presenting the data of Fig. 2 is shown as Fig. 3. The selection and presentation of the $m = 1$

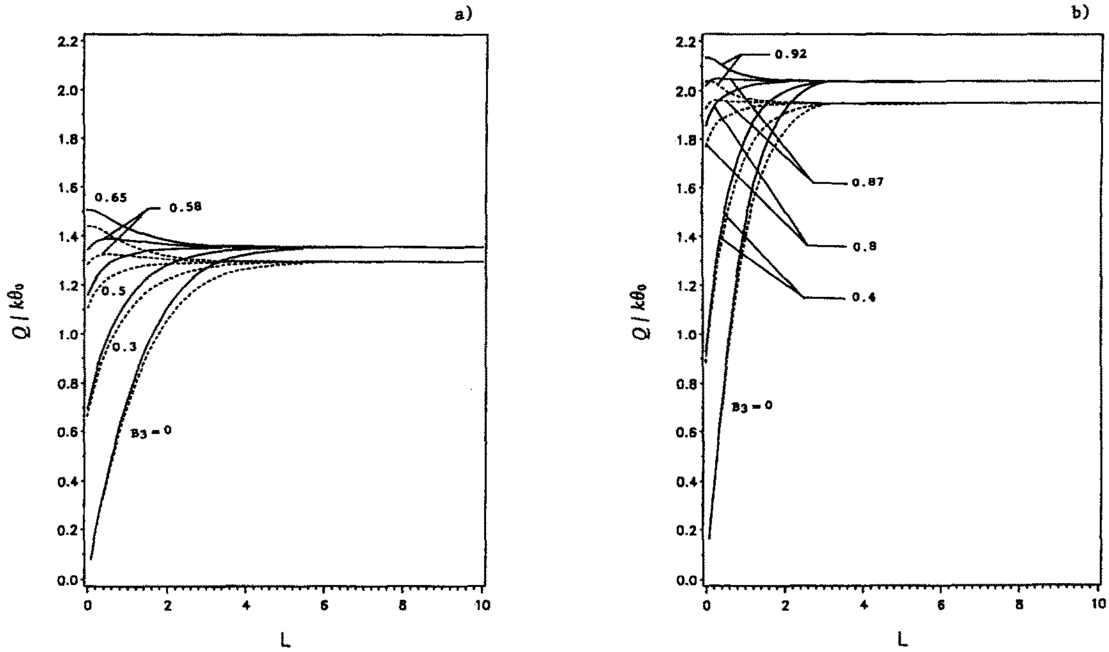

FIG. 4. Heat loss, $Q/k\theta_0$, from a fin vs fin ratio, L, for $B_1 = 1$, and $b = 0.25$ for various values of B_3 and $m = 1$(——) and $m = 3$(– – – –): (a) $B_2 = 0$ and (b) $B_2 = 0.5$.

case will not limit applicability of the results and conclusions since the information available from Fig. 3 is independent of m. Further, L for $0.99(Q/k\theta_0)_{max}$ is used instead of $(Q/k\theta_0)_{max}$ as a matter of convenience because L must be so much longer to get that last 1%. Note that when the values of B_3 are small, L for $0.99(Q/k\theta_0)_{max}$ decreases monotonically as the ratio of B_2/B_1 increases until B_3 approaches a certain value for which the curve varies irregularly. This irregular character begins, for the $B_1 = 0.01$ case, when the value of B_3 is between 0.05 and 0.06.

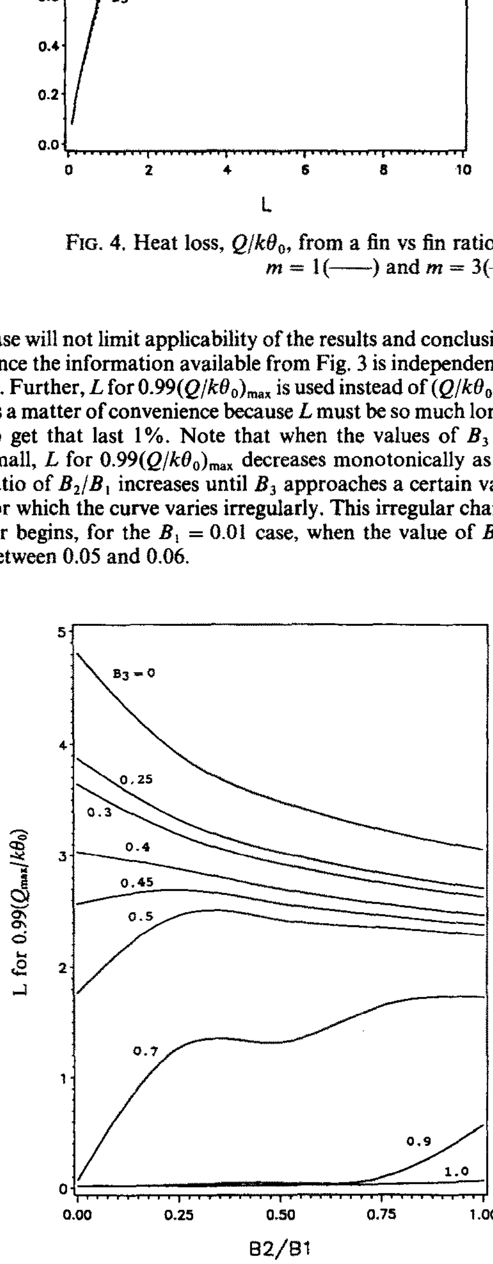

FIG. 5. Optimum fin ratio, L, for $0.99(Q/k\theta_0)_{max}$ dependent upon B_2/B_1 and B_3 (when $B_1 = 1.0$ and $b = 0.25$).

Figure 4 depicts the same type of information as was presented in Fig. 2 but for $B_1 = 1$ (traditionally $B = 1$ indicates that a fin is not justified). Most notable in this figure is that the effect of B_3 on the heat loss disappears when L is approximately 5.

The alternate version of these data is illustrated in Fig. 5. The trend of these curves is similar to the $B_1 = 0.01$ case for small values of B_3. Note that the $B_1 = 1$ case does have certain limited situations where the fin is justified. The difference is that in all cases L for $0.99(Q/k\theta_0)_{max}$ is much shorter when compared with the $B_1 = 0.01$ case. Further, the irregular character begins when B_3 is between 0.4 and 0.45 and as B_3 approaches 1, $L = 0.99(Q/k\theta_0)_{max}$ is almost zero for all values of B_2/B_1 (i.e. $(Q/k\theta_0)_{max}$ occurs at $L = 0$).

The interpretation of the data where the irregular variation occurs ($0.5 < B_3 < 0.9$ in Fig. 5) requires further discussion. If the irregularity is generally upward, Table 1 must be used to determine the fin's usefulness. Table 1 lists the approximate value of B_3 ($=B_w$) which makes the heat loss in the no-fin case larger than that in the infinite length case for given B_1 and B_2. Thus, from Table 1, we can estimate the fin's usefulness for given values of B_1, B_2 and B_3. For example, in the $B_1 = 0.01$ case of Fig. 3 and $B_3 = 0.082$, using Table 1 we see that $B_3 = 0.082$ is between $B_3 = 0.0787$ for $B_2/B_1 = 0.25$ and $B_3 = 0.0863$ for $B_2/B_1 = 0.5$. Thus, the fin is useful for $B_2/B_1 > 0.359$ and not useful for $B_2/B_1 < 0.359$.

CONCLUSIONS

From these results, we can conclude the following:

(1) even though the difference in the values of $Q/k\theta_0$ ($m = 1$) and $Q/k\theta_0$ ($m = 3$) increases as L increases, the trends are the same; thus the resulting conclusions for various m would be approximately the same;

(2) when d^2Q/dL^2 is almost zero, the heat loss from the fin is essentially independent with the fin ratio;

(3) plots of L for $0.99(Q_{max}/k\theta)$ vs B_2/B_1 with B_3 as the parameter is a convenient means of determining fin usefulness; thus for given values of B_3:

Table 1. Values of B_3 which makes the heat loss in case of no-fin larger than that in the case of a fin of infinite length for given values of B_1 and B_2

	B_3 for		
B_2/B_1	$B_1 = 0.01$	$B_1 = 0.1$	$B_1 = 1$
0	0.0704	0.2162	0.5854
0.25	0.0787	0.2438	0.6934
0.5	0.0863	0.2678	0.7702
0.75	0.0932	0.2893	0.8301
1	0.0996	0.3087	0.8796

(a) if L for $0.99(Q/k\theta_0)_{max}$ decreases monotonically as the ratio of B_2/B_1 increases, then the fin is useful for all given values of B_1 and B_2;

(b) if L for $0.99(Q/k\theta_0)_{max}$ varies irregularly as the ratio of B_2/B_1 increases, then a check using Table 1 must be made to determine the fin's usefulness;

(c) if L for $0.99(Q/k\theta_0)_{max}$ is nearly zero as the ratio of B_2/B_1 increases, then the fin is not useful for all values of B_1 and B_2.

REFERENCES

1. M. Avrami and J. B. Little, Diffusion of heat through a rectangular bar and the cooling and insulating effect of fins, *J. Appl. Phys.* **13**, 255–264 (1942).
2. L. C. Burmeister, Triangular fin performance by the heat balance integral method, *ASME J. Heat Transfer* **101**, 562–564 (1979).
3. R. K. Irey, Errors in the one-dimensional fin solution, *ASME J. Heat Transfer* **90**, 175–176 (1968).
4. H. H. Keller and E. V. Somers, Heat transfer from an annular fin of constant thickness, *ASME J. Heat Transfer* **81**, 151–156 (1959).
5. W. Lau and C. W. Tan, Errors in one-dimensional heat transfer analyses in straight and annular fins, *ASME J. Heat Transfer* **95**, 549–551 (1963).
6. A. D. Snider and A. D. Kraus, Recent developments in the analyses and design of extended surfaces, *ASME J. Heat Transfer* **105**, 302–306 (1983).
7. H. C. Unal, The effect of the boundary condition at a fin tip on the performance of the fin with and without internal heat generation, *Int. J. Heat Mass Transfer* **31**(7), 1483–1496 (1988).
8. D. C. Look, Jr. and H. S. Kang, Effects of variation in root temperature on heat lost from a thermally non-symmetric fin, *Int. J. Heat Mass Transfer* **34**(4/5), 1059–1065 (1991).
9. A. Ullmann and H. Kalman, Efficiency and optimized dimension of annular fins of different cross-section shapes, *Int. J. Heat Mass Transfer* **32**, 1104–1110 (1989).
10. A. Brown, Optimum dimensions of uniform annular fins, *Int. J. Heat Mass Transfer* **8**, 655–662 (1965).
11. I. Mikk, Convective fin of minimum mass, *Int. J. Heat Mass Transfer* **23**, 707–711 (1980).
12. A. D. Kraus, Analysis of extended surfaces, *J. Heat Transfer* **110**, 1071–1073 (1988).

Int. J. Heat Mass Transfer. Vol. 35, No. 8, pp. 2060–2065, 1992
Printed in Great Britain
0017-9310/92 $5.00+0.00

Diffusion-thermo and thermal-diffusion effects in transient and steady natural convection from a vertical surface

ZAFER DURSUNKAYA
Ricardo-North America, 645 Blackhawk Drive, Westmont, IL 60559, U.S.A.

and

WILLIAM M. WOREK†
Department of Mechanical Engineering, University of Illinois at Chicago, P.O. Box 4348, Chicago, IL 60680, U.S.A.

(*Received* 11 *April* 1990 *and in final form* 5 *August* 1991)

INTRODUCTION

IN THIS paper, the effect of thermal-diffusion and diffusion-thermo on transient and steady natural convection heat and mass transfer from a vertical surface are investigated numerically. A helium–air mixture was selected as the fluid pair used in the study due to its radically different thermodynamic properties as compared to other fluid pairs. Results showing steady temperature and concentration distributions and the total heat and mass transport from the wall with and without heat and mass transfer coupling are presented. Also, the transient variation of the heat flux from the wall including and neglecting the coupling effects are documented.

The effect of diffusion-thermo and thermal-diffusion on the transport of heat and mass were developed from the kinetic theory of gases by Chapman and Cowling [1]. Hirshfelder *et al.* [2] explained the phenomena and derived the necessary formulae to calculate the thermal-diffusion coefficient and the thermal-diffusion factor for monatomic gases. Although the derivation is restricted to monatomic gases, they found that the error involved with applying the formulae to polyatomic gas mixtures is small.

Hall [3] developed the energy, diffusion and momentum equations for multicomponent systems. He further simplified the equations of motion, energy and diffusion for a steady compressible, boundary layer flow of a binary mixture over a flat plate.

† To whom correspondence should be addressed.